MAT 264: Exercise 1

Submission Deadline: 15 March 2017

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Non-dimensionalization and dimensional analysis

Dimensional analysis is one of the most useful techniques for deducing interesting insights into a physical system. The two exercises deal with basics of dimensional analysis.

Exercise 1 The thrust T developed by a ship propeller in deep water (dimensions of $T = MLT^{-2}$) depends on the radius a of the propeller, the number of revolutions per minute n, the velocity V with which the ship advances, the gravitational constant g, the density ρ, and the kinematic viscosity ν of the water ($[\nu] = L^2T^{-1}$). Show by using the dimensional analysis that

$$\frac{T}{\rho a^2 V^2} = \phi(\frac{an}{V}, \frac{aV}{\nu}, \frac{ag}{V^2}).$$

This exercise shows the use of scaling argument for a "mathematical result".

Exercise 2 Sobolev inequality

Consider an arbitrary smooth function $u : \mathbb{R}^n \to \mathbb{R}^n$ such that $u = 0$ outside an n-dimensional cube of length R. Using scaling arguments, for $n > 2$, derive a condition on q such that there exists a $C > 0$ independent of u for which it holds that

$$\left(\int_{\mathbb{R}^n} |u|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |\nabla u|^2 dx \right)^{\frac{1}{2}}. \tag{1}$$

Hint: Consider $y = \lambda x$ and $w(x) = u(y)$ where $\lambda > 0$ will be chosen later. And notice that the above inequality holds for w. Now for $u(y)$ we derive a similar inequality but for a different C depending on λ and n. Also, note that $\int_{\mathbb{R}^n} |u(x)|^2 dx = \lambda^n \int_{\mathbb{R}^n} |w(y)|^2 dy$.

To extend the result further, let p, n be such that $1 \leq p < n$. Now repeat the above exercise for the case when

$$\left(\int_{\mathbb{R}^n} |u|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |\nabla u|^p dx \right)^{\frac{1}{p}}. \tag{2}$$

Epidemiological Modelling

The emergence of new diseases and their reoccurrence have led to an interest in studying the spreading of infectious diseases. The mathematical modelling is an important tool in analysing the spread, impact and control of such diseases. This exercise is aimed at familiarizing the students with some of the basic models and tools used in the mathematical studies of infectious diseases.

Exercise 3 – Warm up exercise: simple model

We shall consider an elementary model which is inadequate but is simple to analyse and will warm us up for considering further models. Let

$$
\begin{aligned}
S(t) &= \text{ Number of susceptibles at time } t \\
I(t) &= \text{ Number of infectious at time } t \\
N &= \text{ Total number of population} \\
\beta &= \text{ Average number of adequate contacts of a person per unit time}
\end{aligned}
$$

The model is given by

$$
\begin{aligned}
\frac{dS}{dt} &= -\beta IS/N \qquad S(0) = S_0 \geq 0, \\
\frac{dI}{dt} &= \beta IS/N \qquad I(0) = I_0 \geq 0,
\end{aligned}
\tag{3}
$$

with $I_0 + S_0 = N$.

a. Introduce non dimensional variables $i = I/N, s = S/N$ and obtain an ODE for $i(t)$ using the system above. Obtain an explicit solution for this ODE. Is this solution unique?

b. Compute the equilibrium points of the ODE and find out the nature of these points. Conclude that the solutions $i(t), s(t)$ lie between zero and 1 for all time t, if the starting points are between zero and 1.

c. Implement an Euler explicit scheme. Choose $\beta = 0.5, i_0 = 0.1, s_0 = 0.9$ for your numerical tests. Take time steps $h = 5, 1, 0.2, 0.04, 0.008, 0.0016$. What is the condition for the time step to ensure stability of the solution? What are the conditions on the time step for the solutions to respect the equilibrium solutions as bounds? Obtain the error as a function of time step and check the convergence rate. Implement Euler implicit scheme and take the same time steps as for the explicit case. Compute the error and convergence rates. Comment on the results. If time permits it, implement a higher order integration scheme (Runge-Kutta 4) and check the convergence rate. Plot the solution and note the large time asymptotes.

d. Check that $i(t) + s(t) = 1$. Does your numerical scheme preserve this? Show this.

Exercise 4 – Classical SIR model
In addition to the previous variables, we will introduce the removal of people after they are either infection free or succumb to the infection. Let

$$
\begin{aligned}
S(t) &= \text{ Number of susceptibles at time } t \\
I(t) &= \text{ Number of infectious at time } t \\
R(t) &= \text{ Number of people removed from the infectives} \\
N &= \text{ Total number of population} \\
\beta &= \text{ Average number of adequate contacts of a person per unit time} \\
\gamma &= \text{ Removal rate of people per unit time per infective,}
\end{aligned}
$$

and we define, $\sigma = \frac{\beta}{\gamma}$. The classical SIR model consists of

$$\begin{aligned}
\frac{dS}{dt} &= -\beta IS/N & S(0) &= S_0 \geq 0, \\
\frac{dI}{dt} &= \beta IS/N - \gamma I & I(0) &= I_0 \geq 0, \\
\frac{dR}{dt} &= \gamma I & R(0) &= R_0 \geq 0,
\end{aligned} \tag{4}$$

with $I_0 + S_0 + R_0 = N$. For $\gamma = 0$, the model reduces to the previous model.

a. Define the non-dimensional variables $i = I/N, s = S/N, r = R/N$ using N using the total population as the reference quantity. Obtain the non-dimensional equations. Verify that $i(t) + s(t) + r(t) = 1$.

b. Find the equilibrium solutions. Show that $0 \leq i, s, r \leq 1$ and that

$$i(t) + s(t) - \frac{1}{\sigma}\ln(s(t)) = i_0 + s_0 - \frac{1}{\sigma}\ln(s_0), \tag{5}$$

holds and hence is an invariant curve for the differential equation. Plot the level curves.

c. Implement Euler explicit method and plot $i(t)$ and $s(t)$ and interpret the results. For the numerical simulations, you may use $\beta = 0.5, \sigma = 0.5, 1, 2$ and $i_0 = 0.2, s_0 = 0.8$. Try for different values of initial conditions and interpret the results. Check that the numerical solutions indeed respect (5).

d. Verify the following theorem by the numerical experiments. If $\sigma s_0 \leq 1$, then $i(t)$ decreases to zero as $t \to \infty$. If $\sigma s_0 > 1$, then $i(t)$ first increases up to a maximum value $i_{\max} = i_0 + s_0 - \frac{1}{\sigma} - \frac{1}{\sigma}(\ln(\sigma_0))$ and then decreases to zero as $t \to \infty$. The susceptible fraction $s(t)$ is a decreasing function and the limiting value s_∞ is the unique root in $(0, \frac{1}{\sigma})$ of the equation

$$i_0 + s_0 - s_\infty + \frac{1}{\sigma}\ln(s_\infty/s_0) = 0. \tag{6}$$

Interpret the implications of the above theorem.

e. The number of susceptibles s_0 and s_∞ are effectively known by medical tests of the population. Assuming that they are known and i_0 is small, estimate σ.

f. Linearize the equations around the equilibrium points. Compute the eigen values and the eigen vectors of the Jacobian around the equilibrium points. Can you conclude about the nature of these points?

g. Plot the phase portraits for i and s for $\sigma = 0.5$ and 2 and with fixed $\beta = 1$. Describe qualitatively the dynamics of the system. Comment on the following statement: "The final number of susceptibles is as far below the threshold for an epidemic as the initial number was above the threshold".

Exercise 5 – Extension of the model

A general model for epidemiology is MSEIR model with the passively immune class M, the susceptible class S, the exposed class E, the infective class I, and the recovered class R. Quite often, one considers only some of the variables and accordingly gets the name. For example, we will consider a non-dimensional SEIR model

$$
\begin{aligned}
\frac{ds}{dt} &= -\lambda(t)is + \mu - \mu s, & s(0) &= s_0, \\
\frac{de}{dt} &= \lambda(t)is - (\varepsilon + \mu)e, & e(0) &= e_0, \\
\frac{di}{dt} &= \varepsilon e - (\gamma + \mu)i, & i(0) &= i_0, \\
\frac{dr}{dt} &= \gamma i - \mu r, & r(0) &= r_0,
\end{aligned} \tag{7}
$$

with $s + e + i + r = 1$ and the variables s, e, i, r are the non-dimensionalized counterparts of S,E,I,R (for example, $s = \mathrm{S}/N$ with N being the reference population). Moreover, $\lambda, \mu, \varepsilon, \gamma > 0$ are parameters of the model. Note that we have chosen λ to be time dependent. As we increase the number of variables, our model becomes "rich", however, one of the downsides is that we have several parameters for this model. These coefficients are often not known and need to be fitted.

a Discuss a possible interpretation of the model. For example, discuss how the growth of one variable gets affected by others.

b Implement an Euler explicit scheme and perform simulations for the model. Choose $\lambda(t) = \exp(\sin(2t))$ and $\varepsilon, \mu, \gamma = 0.5$. Choose the initial conditions appropriately. What would be a possible interpretation for periodic λ? Interpret the results for your numerical simulations.

c What are the equilibrium solutions? Does the numerical simulations help in guessing the equilibrium solutions and their nature?

Suggested reading

Herbert W. Hethcote, The Mathematics of Infectious Diseases, *SIAM Review Vol. 42, 2000*, pp. 599–653. This has an extensive set of references also.